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Generalised Quaternion Methods in Conformal Geometry

ERIC A. LORD

Department of Applied Mathematics, Indian Institute of Science, Bangalore 560012, *India*

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Abstract

A new approach to Penrose's twistor algebra is given. It is based on the use of a generalised quaternion algebra for the translation of statements in projective five-space into equivalent statements in twistor (conformal spinor) space. The formalism leads to $SO(4, 2)$-covariant formulations of the Pauli–Kofink and Fierz relations among Dirac bilinears, and generalisations of these relations.

1. *Introduction*

The two-one correspondence between the rotations of a sphere, and the subgroup $z \to (\alpha z + \beta)/(-\bar{\beta} z + \bar{\alpha})$, $|\alpha|^2 + |\beta|^2 = 1$, of the group of conformal (circle-preserving) transformations in a plane, through stereographic projection, and the law of combination of these transformations in the plane (essentially multiplication in $SU(2)$), were first clearly established by Cayley (1879). They were, however, implicit in much earlier work (that circles on the sphere are mapped onto circles in the plane by stereographic projection was known to Ptolemy). The correspondence is discussed and references to early work given by Klein (1884). The appropriate algebra for dealing with the group of rotations of a sphere is the quaternion algebra of Hamilton (1844, 1866), which is intimately connected with the above correspondence; the defining relations of the quaternion algebra are in fact identical with those of the Lie algebra of $SU(2)$. In view of these historical facts, it seems remarkable that the two-dimensional matrix representation of quaternion algebra remained unknown until its discovery by Pauli (1927). This work was also the first indication of the fundamentally important physical significance of the two-component representation of the rotation group. Also remarkable is the fact that the Lorentz transformations, and Lorentz-covariant equations, can be very elegantly expressed in terms of quaternions (Klein, 1911; Silberstein, 1912, 1913; Kilmister, 1953, 1955; Rastall, 1964; etc.), which were first propounded sixty years before the advent of relativity.

The correspondence quoted above can be expressed by the statement that the *affine* transformations in three-space that leave the sphere unchanged give rise to a subgroup of the group of conformal mappings in the plane, by stereographic projection. If we consider the group of *projective* transformations that leave the sphere unchanged (which is isomorphic to the *Lorentz* group), we obtain by stereographic projection the group $SL(2, C)$ of Mobius transformations

$$z \to (\alpha z + \beta)/(\gamma z + \delta), \qquad \alpha\delta - \beta\gamma = 1$$

which is just the continuous part of the group of conformal mappings in the plane. (It is also, of course, the group of projective transformations in one *complex* dimension.) More generally, the N-dimensional conformal group can be realised as the group of projective transformations in $(N + 1)$-dimensional space that leave a hypersphere unchanged, through stereographic projection (Klein, 1926; Coxeter, 1936). This correspondence was recognised as having possible implications in physics, by Dirac (1936), who was the first to apply it to the conformal group in Minkowski space. We shall review briefly the salient features of this well-known correspondence between Minkowski space M and a hyperquadric Q in projective five-space P_5.

Denote the homogeneous components of a point in projective five-space by ξ^A $(A = 1, \ldots 6)$ and let Q be the quadric with matrix η_{AB} (the diagonal matrix $(+++--+)$), which we use as a lowering and raising operator for the six-fold indices. Then ξ_A is the polar hyperplane of ξ^A with respect to Q, and the equation of Q is simply

$$\xi^A \xi_A = 0 \tag{1.1}$$

The stereographic projection of Q on to M is given by

$$\left.\begin{aligned} \xi^\mu &= \xi x^\mu && (\mu = 1, \ldots 4) \\ \chi &= \xi x^2 && (x^2 = x_\mu x^\mu) \\ \xi &= \xi^5 + \xi^6 && \\ \chi &= \xi^5 - \xi^6 && \end{aligned}\right\} \tag{1.2}$$

The group of projective transformations on P_5 that preserve Q is $O(4, 2)$, which, through (1.2) is isomorphic to the group of conformal mappings on M. (By a slight modification a correspondence between Q and de Sitter space can be established (Coxeter, 1943; Lord, 1974). We make P_5 into an affine space by specialising $\xi^6 = 0$ as the hyperplane at infinity, and then into a pseudo-Euclidean space by imposing the metric $d\chi_a d\chi^a$ on the inhomogeneous components $\chi^a = \xi^a/\xi^6$ $(a = 1, \ldots 5)$. Then (1.2) induces the $(3 + 2)$ de Sitter metric on M. With $\xi^5 = 0$ as the hyperplane at infinity we get $(4 + 1)$ de Sitter space.)

The exceptional points of Q for which $\xi = 0$ have no image in M, so Q must be regarded as equivalent to Minkowski space *completed* by the inclusion of a space at infinity. In the following, M will denote this completed Minkowski space. The figures in M that are preserved by conformal mappings are the 'Minkowski spheres'. They are hyperquadrics whose asymptotic cones are null cones. Non-null hyperplanes and null cones are regarded as degenerate M-spheres and null planes as degenerate null cones. The M-spheres correspond to intersection of Q by hyperplanes χ_A in P_5. We have M-spheres of one sheet or two, or null cones, according as $\chi^A \chi_A$ is positive, negative or zero. Null cones therefore correspond to hyperplanes *tangential* to Q, so the space at infinity of M has to be regarded as a *null cone*, because $\xi = 0$ is clearly the equation of a hyperplane tangential to Q (to avert misunderstanding, note that the device of regarding the vertex of the null cone at infinity (image of the origin of M under $\xi^6 \to -\xi^6$) as three distinct points, according to whether it is approached from a space-like, future-pointing time-like, or past-pointing time-like direction (Sachs, 1963; Penrose, 1969) is out of place in this context. From the point of view of the projective geometry of P_5, the null cone at infinity is in no way distinguished. It resembles any other null cone.)

The twistor algebra of Penrose (1967) is based on a correspondence between lines in Q (which under (2.1) correspond to *null* lines in M) and points on a quadric B (N in Penrose's terminology) in *complex* projective three-space P_3. Denoting the homogeneous coordinates of a point in P_3 (a 'twistor') by ψ_α ($\alpha = 1, \ldots 4$), we can distinguish two kinds of quadric; *Hermitian* quadrics have equations of the form $\psi^\dagger \beta \psi = 0$ (β Hermitian) and *symmetric* quadrics have equations of the form $\psi x \psi = 0$ (x symmetric). A particular Hermitian quadric B, whose matrix β has signature $(++--)$, is singled out. The quantities $\bar{\psi} = \psi^\dagger \beta$ are the components of the plane which is the polar with respect to B of the point ψ. The equation of B is then simply

$$\bar{\psi}\psi = 0 \tag{1.3}$$

The continuous part of the group of projective transformations in P_3 that preserves B is clearly $SU(2, 2)$. This is the covering group of $SO(4, 2)$, the continuous part of the group of conformal mappings on M. The situation is entirely analogous to the relationship between $SL(2, C)$ the Lorentz group, and the group of conformal mappings in a plane. The purpose of this work is to explore the relationship between P_5 and P_3 (or between Q and B) by methods which are the analogue of *quaternion* methods. Our aim in doing so is to put forward an alternative approach to the geometry of twistors, and to use the twistor algebra as a vehicle for exploring the generalised quaternion methods.

The *physical* significance of the work is at present obscure. The justification for its presentation in a theoretical physics journal rests on the observations that the conformal group on M has been under active investigation for its physical content in recent years (Barut, 1968; Mack & Salam, 1969; Salam & Strathdee, 1969) and that the lower dimensional analogue that led to the concept of spinors was known to mathematicians at least forty years before it played any role in theoretical physics.

In dealing with quantities in projective geometry we make extensive use of Plücker coordinates for lines, and Grassmann coordinates (generalised Plücker coordinates) for planes, three-spaces, etc. (Hodge & Pedoe, 1968).

2. *Algebra*

The Dirac algebra is the Lie algebra of $SO(4, 2)$ (see for example ten Kate (1968)). It can be treated as a generalised quaternion algebra (Lord, 1972a), which we shall call the σ-algebra. The σ-algebra is defined by six 'generalised quaternion elements' σ^A and their conjugates $\bar{\sigma}^A$. The defining relations of the algebra are

$$\sigma_A = (\sigma_a, 1), \qquad \bar{\sigma}_A = (-\sigma_a, 1) \qquad (a = 1, \ldots 5) \tag{2.1}$$

$$\sigma^{(A}\bar{\sigma}^{B)} = \eta^{AB}, \qquad \bar{\sigma}^{(A}\sigma^{B)} = \eta^{AB} \tag{2.2}$$

where η^{AB} is the matrix of Q. We introduce the skewsymmetrised products

$$\left.\begin{aligned} \sigma_{AB} &= \sigma_{[A}\bar{\sigma}_{B]}, & \bar{\sigma}_{AB} &= \bar{\sigma}_{[A}\sigma_{B]} \\ \sigma_{ABC} &= \sigma_{[A}\bar{\sigma}_{BC]}, & \bar{\sigma}_{ABC} &= \bar{\sigma}_{[A}\sigma_{BC]} \end{aligned}\right\} \tag{2.3}$$

In the four-dimensional representation, the sixteen σ_{AB}, 1 are the sixteen base elements of the Dirac algebra, as also are the sixteen σ_A, σ_{ABC}. There exist 4 x 4 matrices β (Hermitian and unitary) and C (skewsymmetric and orthogonal) with the properties

$$\beta\sigma^A\beta = (\bar{\sigma}^A)^\dagger, \qquad C\sigma^A C = -(\sigma^A)^T, \qquad C\beta = -\beta C \tag{2.4}$$

(Note, incidently, that this implies that the generalised quaternion conjugation is in a sense 'charge conjugation'!) The quantities $(1/2)\sigma_{AB}$ and $(1/2)\bar{\sigma}_{AB}$ are infinitesimal generators of two inequivalent irreducible spinor representations of $SO(4, 2)$ (see Lord (1972b), where the N-dimensional generalisation of the σ-algebra is discussed). They are traceless. The six $\sigma^A C$ are skew-symmetric and the ten $\sigma^{ABC}C$ are symmetric. We write their components as

$$\sigma^A{}_{\alpha\beta}, \qquad \sigma^{ABC}{}_{\alpha\beta} \tag{2.5}$$

and the components of $C\bar{\sigma}_A$ and $C\bar{\sigma}_{ABC}$ will be written

$$\sigma_A{}^{\alpha\beta}, \qquad \sigma_{ABC}{}^{\alpha\beta} \tag{2.6}$$

The following 'duality' properties hold

$$\left.\begin{aligned} \sigma^{ABC} &= (i/6)\epsilon^{ABCDEF}\sigma_{DEF} & &\text{(selfdual)} \\ \bar{\sigma}^{ABC} &= -(i/6)\epsilon^{ABCDEF}\bar{\sigma}_{DEF} & &\text{(anti-selfdual)} \end{aligned}\right\} \tag{2.7}$$

$$\sigma^A{}_{\alpha\beta} = (1/2)\epsilon_{\alpha\beta\gamma\delta}\sigma^{A\gamma\delta} \tag{2.8}$$

Associated with every x_A is a skewsymmetric (4 x 4) matrix, with every $x_{AB} = -x_{BA}$ a traceless matrix, and with every completely skewsymmetric selfdual (anti-selfdual) x_{ABC} a symmetric matrix $x^{\alpha\beta}$ $(x_{\alpha\beta})$:

$$\left.\begin{array}{lll} x_{\alpha\beta} = x_A\,\sigma^A_{\alpha\beta} & (x_{\alpha\beta} = -\,x_{\beta\alpha}), & x_A = (1/4)x_{\alpha\beta}\sigma^{\alpha\beta}_A \\ x^\beta_\alpha = x^{AB}\sigma_{AB}{}_\alpha{}^\beta & (x^\alpha_\alpha = 0), & x_{AB} = -(1/8)x^\beta_\alpha\sigma_{AB\beta}{}^\alpha \\ x^{\alpha\beta} = x^{ABC}\sigma^{\alpha\beta}_{ABC} & (x^{\alpha\beta} = x^{\beta\alpha}), & x^{ABC} = -(1/48)x^{\alpha\beta}\sigma^{ABC}_{\alpha\beta} \\ & & (x^{ABC}\ \text{selfdual}) \\ x_{\alpha\beta} = x_{ABC}\sigma^{ABC}_{\alpha\beta}, & (x_{\alpha\beta} = x_{\beta\alpha}), & x_{ABC} = -(1/48)x_{\alpha\beta}\sigma^{\alpha\beta}_{ABC} \\ & & (x_{ABC}\ \text{anti-selfdual}) \end{array}\right\} \quad (2.9)$$

We list below the additional properties of the σ-algebra that we shall require:

$$\left.\begin{array}{ll} (1/2)\sigma_{AB}{}^\alpha_\beta\,\sigma^{AB\gamma}{}_\delta = (\delta^\alpha_\beta\delta^\gamma_\delta - 4\delta^\gamma_\beta\delta^\alpha_\delta) \\ (1/12)\sigma^{\alpha\beta}_{ABC}\sigma^{ABC}_{\gamma\delta} = -4\delta^\alpha_{(\gamma}\delta^\beta_{\delta)} \\ \sigma^{\alpha\beta}{}_A\,\sigma^A_{\gamma\delta} = -4\delta_{[\gamma}{}^\alpha\delta^\beta_{\delta]} \\ \sigma_{A\,\alpha\beta}\sigma^A{}_{\alpha\beta} = -2\epsilon_{\alpha\beta\gamma\delta} \end{array}\right\} \quad (2.10)$$

$$\sigma^{\alpha\beta}_{[A}\sigma_{B]\gamma\delta} = 2\sigma_{AB}{}_{[\gamma}^{[\alpha}\delta^{\beta]}_{\delta]} \quad (2.11)$$

For any two traceless matrices x and y (with x_{AB} and y_{AB} given by (2.9)),

$$\left.\begin{array}{rl} ix^{ABCD}y_{CD} &= 4x_E{}^{(A}y^{B)E} + 1/8x^\beta_\alpha y^\delta_\gamma(\sigma^{A\,\gamma\alpha}\sigma^B_{\beta\delta} - 2\eta^{AB}\delta^\alpha_{[\beta}\delta^\gamma_{\delta]}) \\ &= 4x_E{}^{[A}y^{B]E} - 1/8x^\beta_\alpha y^\delta_\gamma(\sigma^{AB}{}^\alpha_\delta\delta^\gamma_\beta) \end{array}\right\} \quad (2.12)$$

Where we have introduced the notation

$$x^{ABCD} = (1/2)\epsilon^{ABCDEF}x_{EF} \quad (2.13)$$

The relation (2.12) is proved as follows. From the results of Lord (1973a) it is a simple matter to obtain an expression for the trace of $\sigma_{AB}\sigma_{CD}\sigma_{EF}\sigma_{GH}$. Multiply this expression by $x^{CD}y^{GH}$. Then do the same for the trace of $\sigma_{AB}\sigma_{CD}\sigma_{GH}$. The two relations (2.12) can then be easily derived. The proof of (2.11) goes as follows. For any pair of *arbitrary* skewsymmetric matrices x and y,

$$\begin{aligned} \sigma_{AB}{}^\beta_\alpha\,x_{\beta\gamma}y^{\gamma\alpha} = \operatorname{tr}\sigma_{AB}xy = -\,x_Cy_D\operatorname{tr}\sigma_{AB}\sigma^{CD} &= 8x_{[A}y_{B]} \\ &= (1/2)x_{\alpha\beta}y^{\gamma\delta}\sigma^{\alpha\beta}_{[A}\sigma_{B]\gamma\delta} \end{aligned}$$

3. *Correspondences Between P_5 and P_3*

The one-one correspondence between (complex) points in P_5 and null polarities (line complexes) in P_3 is an immediate consequence of (2.9). With every point x^A of P_5 we can associate a skewsymmetric (4×4) matrix $x_{\alpha\beta}$. This matrix will be the set of Plücker coordinates of a line in P_3 (i.e. will have the form $\chi_{[\alpha}\psi_{\beta]}$) if and only if

$$x_{[\alpha\beta}x_{\gamma\delta]} = 0 \quad (3.1)$$

or, equivalently, defining the dual coordinates $\tilde{x}$ by the matrix

$$x^{\alpha\beta} = 1/2\epsilon^{\alpha\beta\gamma\delta} x_{\gamma\delta} \tag{3.2}$$

the condition for the matrix x to represent a line in P_3 is

$$x_{\alpha\beta}x^{\alpha\beta} = 0 \tag{3.3}$$

($\mathrm{tr}\, x\tilde{x} = 0$), which is just

$$x^A x_A = 0 \tag{3.4}$$

This establishes the one-one correspondence between (complex) point on Q, and lines in (complex) P_3.

If the point in P_5 is real, (2.4) gives

$$\tilde{x} = \beta x^\dagger \beta \tag{3.5}$$

For a real point on Q this is just the condition for its line image in P_3 to lie entirely in B. *We have a one-one correspondence between real points on Q and generators of B.*

The second line of (2.9) establishes a one-one correspondence between null polarities in P_5 and traceless mappings on P_4. If a given $x^{AB} = -x^{BA}$ in P_5 is to represent a line (i.e. is to have the form $x^{[A}x^{B]}$), then

$$x^{[AB}x^{CD]} = 0 \tag{3.6}$$

or, equivalently,

$$x^{AB}x_{ABCD} = 0 \tag{3.7}$$

where the four-index symbol (set of 'dual' coordinates of the line) is defined by (2.13). If we raise all its indices, it becomes the set of Grassmann coordinates for the polar three-space of the line with respect to Q. We wish to express (3.7) as a condition on the traceless 4 x 4 matrix x_α^β. Setting $x = y$ in (2.12) and skewsymmetrising on AB we see immediately that (3.7) is $\mathrm{tr}\, \sigma_{AB}x^2 = 0$. Multiplying this by an arbitrary y^{AB} we obtain that $\mathrm{tr}\, yx^2 = 0$ for arbitrary traceless y. Hence x^2 is a *multiple of the unit matrix.* This is the required condition for x^{AB} to be Plücker coordinates of a line. Consider the special case $x^2 = 0$. Since x^2 must be a multiple of the unit matrix, a sufficient condition for this is $\mathrm{tr}\, x^2 = 0$, or

$$x_\alpha^\beta x_\beta^\alpha = 0 \tag{3.8}$$

But this is just

$$x^{AB}x_{AB} = 0 \tag{3.9}$$

This latter equation is the condition for the line x^{AB} to *intersect Q. We have a one-one correspondence between lines that do not intersect Q and traceless involutions on P_3.*

The condition for a line to lie entirely in Q is

$$x_{AB}x^{BC} = 0 \tag{3.10}$$

Application of the first relation (2.12) in conjunction with (3.7) shows that this is equivalent to

$$x_\alpha^\beta x_\gamma^\delta \sigma^{A\gamma\alpha} \sigma^B_{\beta\delta} = 0$$

or $x_\alpha^\beta x_\gamma^\delta y^{\gamma\alpha} z_{\beta\delta} = 0$ for arbitrary skewsymmetric y and z, or

$$x^\beta_{[\alpha} x^\delta_{\gamma]} = 0 \tag{3.11}$$

Now (3.12) is just the condition for x_α^β to have the form

$$x_\alpha^\beta = \chi^\beta \psi_\alpha \tag{3.12}$$

(with $\chi\psi = 0$ because x is traceless). We have established *a one-one correspondence between (complex) lines in Q and structures in P_3 consisting of a plane (χ) and a point (ψ) on it.*

For a *real* line in P_5,

$$x = -\beta x^\dagger \beta \tag{3.13}$$

For real lines in Q this gives $\chi = i\psi^\dagger\beta = i\bar{\psi}$ so

$$x_\alpha^\beta = i\bar{\psi}^\beta \psi_\alpha, \qquad (\bar{\psi}\psi = 0) \tag{3.14}$$

The plane χ is now tangential to B and ψ is its point of contact. This is the one-one correspondence between real *lines in Q and points on B*, that is fundamental in twistor algebra. Equivalently, null lines in M are in one-one correspondence with null twistors. Note that *the Plücker coordinates of a real line in Q are the fifteen Dirac bilinears $i\bar{\psi}\sigma^{AB}\psi$ constructed from its null twistor.*

The general condition, in P_5, for a line to pass through a given point, is

$$x^{[A} x^{BC]} = 0 \tag{3.15}$$

By means of the σ-algebra methods, the corresponding relation in P_3 can be shown to be

$$x^{\rho(\alpha} x_\rho^{\beta)} = 0, \qquad x_{\rho(\alpha} x_{\beta)}^\rho = 0 \tag{3.16}$$

If the line x^{AB} does *not* intersect Q, the matrix x_ρ^β will be non-singular. Multiplying by the first expression by x_β^σ and making use of $x_\rho^\beta x_\beta^\sigma = k\delta_\rho^\sigma$, we get

$$kx^{\alpha\sigma} = x_\rho^\alpha x_\beta^\sigma x^{\rho\beta} \tag{3.17}$$

(The second equation (3.16) leads to the same result.) The null polarity that corresponds to the point in P_5 is *invariant* under the involution that corresponds to the line in P_5, if and only if the point is on the line.

We pass immediately to the special case when the point is on Q. The condition for a line to pass through a given point on Q is that the point shall be on the polar three-space of the line or, equivalently, the line shall lie in the polar hyperplane of the point:

$$x_A x^{AB} = 0 \tag{3.18}$$

Equations (3.15) and (3.17) in conjunction are easily seen, by application of the σ-algebra methods, to be equivalent to

$$x^{\alpha\rho}x_{\rho}^{\beta} = 0, \qquad x_{\alpha\rho}x_{\beta}^{\rho} = 0 \tag{3.19}$$

In particular, the condition for a *real line* $i\bar{\psi}^{\beta}\psi_{\alpha}$ in Q to pass through a *real point* $x_{\alpha\beta}$ ($\bar{x} = \beta x^{\dagger}\beta$) on Q is, from (3.19)

$$x_{[\alpha\beta}\psi_{\gamma]} = 0 \tag{3.20}$$

A real point in Q lies on a real line in Q if and only if the (line) image in B of the point passes through the (point) image in B of the line.

The general condition for a pair of lines x^{AB} and y^{AB} in P_5 to intersect is

$$x^{[AB}y^{CD]} = 0 \tag{3.21}$$

which, through the first equation (2.12) in conjunction with (2.11), leads immediately to the conclusion that the two lines intersect if and only if, for the associated traceless matrices x and y, $(xy + yx)$ is a multiple of the unit matrix. For a pair of lines in Q to intersect, we have the simpler condition

$$x^{AB}y_{AB} = 0 \tag{3.22}$$

(To prove this, note that it can be written $x^{[AB}y^{CDEF]} = 0$ which is the condition for the line x to intersect the polar three-space of y. Thus (3.22) holds if and only if there is a point on x that is in the polar of y. Equivalently, y is in the polar of a point on x. This polar is a hyperplane tangential to Q. Now, a line in Q that lies in a tangent hyperplane necessarily passes through the point of tangency. (The equivalent statement in M, that a null line in a given null cone passes through the vertex, is self-evident.) Hence, y passes through a point on x. This completes the proof that (3.22) is a necessary and sufficient condition for two lines in Q to intersect.) In terms of P_3, (3.22) is just $x_{\alpha}^{\beta}y_{\beta}^{\alpha} = 0$. For two *real* lines in Q, with

$$x_{\alpha}^{\beta} = i\bar{\chi}^{\beta}\chi_{\alpha}, \qquad y_{\alpha}^{\beta} = i\bar{\psi}^{\alpha}\psi_{\beta}$$

We see that (3.22) is equivalent simply to

$$\bar{\chi}\psi = 0 \tag{3.23}$$

A necessary and sufficient condition for a pair of real lines in Q to intersect is that their (point) images in B lie on a generator of B. This generator is of course the image of the point of intersection of the lines in Q. The condition (3.23) for the intersection of two null lines in M is of course one of the fundamental relations of the twistor algebra.

The Grassmann coordinates of a *plane* in P_5 constitute a completely skew-symmetric quantity x^{ABC} satisfying

$$x^{AB[C}x^{DEF]} = 0 \tag{3.24}$$

or, equivalently,

$$x^{ABC}\tilde{x}_{CDE} = 0 \tag{3.25}$$

Where $\tilde{x}_{ABC} = (i/6)\epsilon_{ABCDEF}x^{DEF}$ are the dual coordinates. Through (2.9) we can associate x_{ABC} with *two symmetric* 4 x 4 matrices, $x^{\alpha\beta}$ and $x_{\alpha\beta}$, which specify its selfdual and anti-selfdual parts. The properties of the σ-algebra can be employed to show that (3.25) is

$$x_{\alpha\rho}x_{\gamma\sigma}\epsilon^{\rho\sigma\beta\delta} - x^{\beta\rho}x^{\delta\sigma}\epsilon_{\rho\sigma\alpha\gamma} = (x_{\gamma\rho}x^{\rho\beta})\delta^{\delta}_{\alpha} - (x_{\alpha\rho}x^{\rho\delta})\delta^{\beta}_{\gamma} \qquad (3.26)$$

Contraction of this relation shows that $x_{\alpha\rho}x^{\rho\beta}$ is a multiple of δ^{β}_{α}. If it is non-zero, $x^{\alpha\beta}$ can be taken to be the inverse of $x_{\alpha\beta}$. The right-hand side of (3.26) then vanishes and (3.26) becomes a statement of the *unimodularity* of $x_{\alpha\beta}$.

Consider now the degenerate case, when $x_{\alpha\rho}x^{\rho\beta} = 0$. Since (3.26) holds, $x_{\alpha\rho}x^{\rho\alpha} = 0$ is a sufficient condition for this. But a simple application of the properties of the σ-algebra shows that this is

$$x_{ABC}x^{ABC} = 0 \qquad (3.27)$$

This means that the plane and its polar with respect to Q have at least one point in common. We therefore have a one-one correspondence between non-singular quadrics $x_{\alpha\beta}$ in P_3, and pairs of non-intersecting mutually polar planes in P_5.

When the plane does not intersect its polar, we have a pair of degenerate quadrics for which $x_{\alpha\rho}x^{\rho\beta} = 0$. Multiplying (3.26) by $x_{\mu\beta}$ in this case gives

$$x_{\mu\beta}x_{\alpha\rho}x_{\gamma\sigma}\epsilon^{\rho\sigma\beta\delta} = 0$$

So the rank of $x_{\alpha\beta}$ is only 2, and it therefore has the form

$$\psi_\alpha\psi_\beta + \chi_\alpha\chi_\beta \qquad (3.28)$$

The matrix $x^{\alpha\beta}$ has an analogous form.

An intersecting special case arises for planes that are self-polar with respect to Q. Such a plane is necessarily complex, and the self-polarity can be expressed as selfduality *or* anti-selfduality of x_{ABC}. Thus self-polar planes fall into two classes. For an anti-selfdual plane $x^{\alpha\beta} = 0$ and (3.26) is just

$$x_{\alpha[\beta}x_{\gamma]\delta} = 0$$

which is the condition for $x_{\alpha\beta}$ to have the form $x_{\alpha\beta} = \psi_\alpha\psi_\beta$. For an *anti-selfdual* plane therefore,

$$x^{\alpha\beta} = 0, \qquad x_{\alpha\beta} = \psi_\alpha\psi_\beta \qquad (3.29)$$

The complex conjugate of an anti-selfdual plane is *selfdual* and has

$$x^{\alpha\beta} = -\bar{\psi}^\alpha\bar{\psi}^\beta, \qquad x_{\alpha\beta} = 0 \qquad (3.30)$$

Self-polar planes in P_5 are in two-one correspondence with points in P_3 (a self-polar plane and its complex conjugate corresponding to the *same* point).

Consider now the *real* planes in P_5. The reality of x_{ABC} is easily shown to be equivalent to

$$x^{\alpha\beta} = -\beta^{\alpha\dot{\gamma}}(x^{\dagger})_{\dot{\gamma}\dot{\delta}}\beta^{\dot{\delta}\beta} \tag{3.31}$$

In the general case (when the plane does not intersect its polar and therefore $x^{\alpha\beta}$ is the inverse of $x_{\alpha\beta}$), this is the condition for the quadric $x_{\alpha\beta}$ in P_3 to be self-polar with respect to B. We have a *one-one correspondence between non-intersecting mutually polar pairs of (real) planes in P_5, and quadrics self-polar with respect to B.* On the other hand, for a real plane in P_5 that does intersect its polar, $x_{\alpha\beta}$ has the form (3.28), equation (3.31) gives

$$x^{\alpha\beta} = -\bar{\psi}^{\alpha}\bar{\psi}^{\beta} - \bar{\chi}^{\alpha}\bar{\chi}^{\beta} \tag{3.32}$$

and $x_{\alpha\beta}x^{\alpha\beta} = 0$ gives

$$\bar{\psi}\psi = \bar{\chi}\chi = \bar{\psi}\chi = 0 \tag{3.33}$$

We have a *one-one correspondence between pairs of mutually polar (real) planes in P_5 that intersect in a point* (note that this point is necessarily on Q), *and point pairs in B.* Each such point pair lies on a *generator* of B, which is the image of the point of intersection of the two planes.

In the special case of a pair of real mutually polar planes intersecting in a *line* (this line is necessarily in Q), (3.28) becomes just

$$x_{\alpha\beta} = \psi_{\alpha}\psi_{\beta} \tag{3.34}$$

and we obtain a single point in B as the image of the pair of planes. This point is of course the image of the line of intersection of the two planes. Incidentally, we see from this that, *through any (real) line in Q there is one and only one pair of mutually polar (real) planes.* A real mutually polar pair of planes is characterised as follows: if x_{ABC} is specified by $x_{\alpha\beta}$, $x^{\alpha\beta}$, then its polar is specified by $ix_{\alpha\beta}$, $-ix^{\alpha\beta}$.

4. *Identities Satisfied by Dirac Bilinears*

Various relations between products of the scalar, vector, tensor, pseudovector and pseudoscalar formed from a spinor (or a pair of spinors) are well known. They are the Pauli-Kofink and Fierz relations (Pauli, 1936; Kofink, 1937, 1940; Fierz, 1936). The σ-algebra enables $SO(4, 2)$ covariant relations between bilinears constructed from a pair of twistors to be obtained, which contain the Pauli-Kofink and Fierz relations, and generalisations of them.

Let ψ and χ be two twistors and construct the sixteen bilinears

$$s = \bar{\psi}\chi, \qquad s_{AB} = i\bar{\psi}\sigma_{AB}\chi \tag{4.1}$$

Put $x = y = \chi\bar{\psi} - (1/4)s$ in (2.12) and we obtain

$$s^{EA}s_{E}{}^{B} = s^2\eta^{AB} \tag{4.2}$$

$$s^{ABCD}s_{CD} = -4ss^{AB} \tag{4.3}$$

Where $s^{ABCD} = (1/2)\epsilon^{ABCDE}s_{DE}$. These concise identities become more familiar, but more cumbersome, upon setting $\sigma^{A} = (i\gamma^5\gamma^{\mu}, \gamma^5, 1)$,

$\bar{\sigma}A = (-i\gamma^5\gamma^\mu, -\gamma^5, 1)$ and writing them in four-component notation. It is very easy, by employing the σ-algebra, to obtain similar expressions with four 'twistors' instead of two, but they are complicated.

Equation (4.3) can be re-expressed in different but equivalent ways. Contraction with the alternating symbol gives

$$\begin{aligned}-4ss_{ABCD} &= (1/2)\epsilon_{ABCDEF}s^{EFGH}s_{GH} = (1/4)\epsilon_{ABCDEF}\epsilon^{EFGHIJ}s_{IJ}s_{GH}\\ &= 12\delta^{GHIJ}_{[ABCD]}s_{IJ}s_{GH}\end{aligned}$$

Hence

$$ss_{ABCD} = -\ 3s_{[AB}s_{CD]} \tag{4.4}$$

The right-hand side here is equal to $-3s_{A[B}s_{CD]}$. Contraction of BCD with the alternating symbol gives finally

$$-3ss^{[AB}\delta^{C]}_{E} = s^{ABCD}s_{ED} \tag{4.5}$$

It is curious that (4.5) looks like a generalisation of (4.3), but is, in fact, just a reformulation of the same expression.

If now s_{AB} are constructed from a *single* twistor with $s = 0$, then (4.3) reduces to the condition for s_{AB} to be Plücker coordinates of a line in P_5, and (4.2) reduces to the statement that this line is in Q. In this case, (4.2) and (4.3) are a highly redundant set of identities. Writing $s^\mu = s^{\mu 6}$ and $s^5 = s^{56}$, if $s^5 \neq 0$ they can all be derived from the minimal set

$$\left.\begin{aligned} s_\mu s^\mu &= -\ s^5_\mu s^{\mu 5} = (s^5)^2\\ s^5 s^{\mu\nu} &= 2s^{[\nu}s^{\mu]5}\end{aligned}\right\} \tag{4.6}$$

It follows that if $s_5 \neq 0$ the entire s_{AB} can be built up from the knowledge of

$$y^\mu = s^\mu/s^5, \qquad l^\mu = (s^\mu + s^{\mu 5})/s^5 \tag{4.7}$$

which transform as vectors under the conformal transformations in M induced by $SO(4, 2)$ transformation in P_5. They satisfy

$$l^\mu l_\mu = 0, \qquad y^\mu y_\mu = y^\mu l_\mu = 1 \tag{4.8}$$

and therefore specify a null line

$$x^\mu = y^\mu + \alpha l^\mu \tag{4.9}$$

The coordinates of the image in Q of the general point on this null line are

$$\left.\begin{aligned}\xi^\mu &= y^\mu + \alpha l^\mu\\ \xi^5 &= (1+\alpha)s^5\\ \xi^6 &= -\ \alpha s^5\end{aligned}\right\} \tag{4.10}$$

It is easy to verify that the Plücker coordinates of the line in Q defined by (4.10) are just s_{AB}. The exceptional null lines that cannot be represented in the form (4.7) with a vector y^μ satisfying (4.8) are those on the light cone at infinity and those on the generators of one-sheeted M-spheres centred at the origin ($x^2 = -\kappa^2$) (equivalently, those in null planes through the origin).

5. *Discrete Conformal Transformations*

The full group of transformations in P_5 that preserve Q is $O(4, 2)$. The 'reflections' induce reflections in M and *inversions* in M-spheres. The full group $O(4, 2)$ induces the full conformal group in M, which is generated by

$$\left.\begin{array}{ll}\text{(i)} & \text{Poincaré transformations including parity and time reversal.}\\ \text{(ii)} & \text{Dilatations.}\\ \text{(iii)} & \text{The special inversion } x^\mu \to x^\mu/x^2 \text{ in the Minkowski sphere}\\ & x^2 = 1.\end{array}\right\} \tag{5.1}$$

We wish to discover the image of this complete group in P_3—i.e. the enlargement of $SU(2, 2)$ that corresponds to the full conformal group.

We noted that $(1/2)\sigma_{AB}$ and $(1/2)\bar{\sigma}_{AB}$ are generators of two inequivalent irreducible $SU(2, 2)$ representations (S and $\bar{S}$) of $SO(4, 2)$. From (2.4) we can show that, if ψ is a twistor transforming according to S, then the 'charge conjugate twistor'

$$\psi^c = C\bar{\psi} \tag{5.2}$$

Transforms according to $\bar{S}$. We have

$$s_{AB} = i\bar{\psi}\sigma_{AB}\psi = i\bar{\psi}^c\bar{\sigma}_{AB}\psi^c \tag{5.3}$$

Since the matrices $\bar{\sigma}_{AB}$ differ from σ_{AB} by a change of sign when one of the indices is equal to 6, we can deduce that the reflection $\xi^6 \to -\xi^6$ (corresponding to the special inversion (5.1(iii)) is represented in the twistor space by 'change conjugation':

$$\psi \to \psi^c \tag{5.4}$$

In a similar manner we can show that inversion $\xi^\mu \to -\xi^\mu$ in the 1234 subspace of P_5 (corresponding to 'PT' transformation in M) is represented by

$$\psi \to \gamma^5\psi \tag{5.5}$$

and that the reflection $\xi^2 \to -\xi^2$ is represented by complex conjugation

$$\psi \to \psi^* \tag{5.6}$$

Now, any element of $O(4, 2)$ can be generated from products of

$$\left.\begin{array}{lll}\text{(i)} & \text{Elements of } SO(4,2) & \\ \text{(ii)} & \xi^\mu \to -\xi^\mu & (\mu = 1 \ldots 4)\\ \text{(iii)} & \xi^2 \to -\xi^2 & \end{array}\right\} \tag{5.7}$$

So the full conformal group is represented in P_3 by products of the operations

$$\left.\begin{array}{ll}\text{(i)} & \text{Elements in } SU(2,2).\\ \text{(ii)} & \text{Multiplication by } \gamma_5.\\ \text{(iii)} & \text{Complex conjugation.}\end{array}\right\} \tag{5.8}$$

The projective transformations in complex P_3 consist of transformations $SL(4, C)$ (up to a factor) and complex conjugation. The most general transformation that preserves the quadric B consists of

$$\psi \to \psi^*, \qquad \psi \to P\psi \tag{5.9}$$

where P is a non-singular complex matrix satisfying

$$P^\dagger \beta P = k\beta \tag{5.10}$$

with k an arbitrary complex number. Since P is to be specified only up to a factor it can be chosen unimodular. The determinant of (5.10) then shows that $k = \pm 1$ or $\pm i$. The complex conjugate of (5.10) shows that it cannot be $\pm i$. For the case $k = +1$, (5.10) states that P belongs to $SU(2, 2)$. Since $\beta\gamma_5 = -\gamma_5\beta$, the most general solution of (5.10) with $k = -1$ is $P = \gamma_5 S$ where S belongs to $SU(2, 2)$. Thus the complete group of projective transformations in P_3 that preserves B is precisely (5.8).

6. *Conclusions*

We have shown that a new, manifestly $O(4, 2)$-covariant formulation of twistor algebra is possible, based on Plücker and Grassmann coordinates in projective geometry, in conjunction with the generalised quaternion algebra. The subject appears to be considerably clarified by these methods and new results are easily obtained. It is hoped that this will stimulate a renewed interest in twistor algebra, which contains some very elegant geometry. The σ-algebra methods can be easily modified (by defining $\sigma_6 = i$) to deal with the well-known correspondence between real lines in P_3 and real points on a quadric with signature $(3 + 3)$ in P_5, and can be generalised to higher dimensions by employing the results of Lord (1973b). This aspect remains to be investigated.

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